ON BALLISTICALLY CLOSED REGIONS

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INTRODUCTION

That there are no oscillatory solutions to the equation

$$\frac{d^2x}{dt^2} = a^2x \qquad ,$$

where a is a real constant, is surely one of the oldest qualitative results about ordinary differential equations. By an oscillatory function, we mean any function which crosses the t-axis more than once. We may put this in a slightly different way. If non-negative boundary values be assigned for two values of t, the solution is non-negative at the values of t between these end points.

In this report we extend this result to systems of ordinary, second-order, linear, homogeneous equations having constant coefficients. In the first investigations a vector was called non-negative when all of its components were so. Necessary and sufficient conditions upon a matrix A were found which insured that solutions to the vector equation

$$\text{(E)} \qquad \frac{d^2x}{dt^2} = Ax$$

remained (in any interval of the t-axis) non-negative between non-negative boundary values. But as the treatment became more geometrical, so the results were found to be more comprehensive. Moreover, the extension to the case of vector equations in Hilbert space, with A a bounded linear transformation, proved to be relatively trivial.

The report commences with an introduction to the problem from a somewhat more applied standpoint which leads to the notion of ballistically closed sets. These are closed sets with an interior point having the property that the arc of any solution joining two points in the set lies wholly within the set. The remainder of the first chapter is concerned with explanations of notation and terminology and closes with a result about certain entire functions of real, bounded, linear transformations.

The second chapter is devoted to a discussion of the solvability of boundary-value problems associated with equation (E).

The third chapter commences with a succession of lemmas which furnish necessary conditions upon the set K, and the transformation A, in order that K be ballistically closed with respect to equation (E).

Following this, we define linear transformations which shrink or expand a convex set. With some slight stipulations the intuitive physical picture of the process gives the correct meaning. The principal result of the report is then proven, viz.

In order that a closed set K, having an interior point, be ballistically closed with respect to equation (E), it is necessary and sufficient that:

(a) K is convex.

(b) There is at least one point $\bar{x}$ of K for which

$$A\bar{x} = 0 \quad .$$

Thus, after a trivial translation, K contains the origin.

(c) The transformation $I + s^2A$ expands K for each choice of the real parameter s .[1)]

Following this, in the fourth chapter, we consider the problem in an n-dimensional Euclidean space. The necessary and sufficient conditions upon the matrix A, which

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1) In (c) it is assumed that the translation mentioned in (b) has already been performed.

insure that $I + s^2 A$ expands a convex set K are determined for the cases in which: K is the first 2^n-tant (i.e., the generalized first quadrant); K is the n-dimensional simplex with one vertex at the origin; K is a hyperspherical or hyperellipsoidal body centered at the origin; K is a hyperconical convex body with its vertex at the origin.

In closing, it is my pleasant task to acknowledge the invaluable guidance and kindness of my doctor-father, Professor Charles Loewner. His many friends will best appreciate my debt. I also wish to express my gratitude to my wife and to Mrs. George Feigen for their assistance in the preparation of the manuscript.

CHAPTER I

1. Let us consider a moving particle of unit mass which is subjected only to a constant downward gravitational force, its position being referred to a moving Cartesian coordinate system introduced in the following fashion: If z represents the elevation above the fixed reference plane in which fixed x, y axes are introduced, let

$$z_1 = z + \frac{1}{2} gt^2$$

$$x_1 = x$$

$$y_1 = y \quad ,$$

where g is the gravitational constant and t represents the time variable. The differential equations of motion of the particle are

$$\frac{d^2x_1}{dt^2} = 0$$

$$\frac{d^2y_1}{dt^2} = 0$$

$$\frac{d^2z_1}{dt^2} = 0 \quad .$$

If a particular path of motion is known to pass through the points (x_1 , y_1 , z_1) and (x_1' , y_1' , z_1') at times t and t' , respectively, then that part of the path which is traversed between these times is just the straight line segment joining these two points. Any convex set K , fixed with respect to the (x_1 , y_1 , z_1) system, thus has the property that any segment of a path of motion of the differential system whose end-points lie within or on the boundary of K lies wholly within K . Moreover, this property is enjoyed only by convex sets.

To continue, let us imagine a slightly more complicated system of equations such as

$$\frac{d^2x}{dt^2} = x$$

$$\frac{d^2y}{dt^2} = y$$

$$\frac{d^2z}{dt^2} = z \quad .$$

As before, suppose a particular path of motion is known to pass through the points P and P' at times t and t', respectively. Assume P and P' lie on a common ray from the origin. In this case the part of the path traversed between these times is not necessarily just the segment joining the two points, but may rather extend past that end point which is nearest the origin. To mark this difference, we call any part of a particular path of motion which is traversed between two fixed times a <u>path segment</u> joining the points occupied at those times. The distinction shows that, for this latter differential system, there are convex sets which do not have the property that the path segment joining two points on the same ray lies wholly within the convex set. In fact, it is easy to see that if, and only if, the convex set has the origin as an interior point or as a boundar point, then such path segments will lie wholly within the convex set.

2. More generally, we shall say that a closed set K having an interior point is ballistically closed with respect to a given second-order differential system if the following two conditions are satisfied for any pair of points P, P' of K (which may be a single point taken twice):

(a) If t and t' be any two distinct times, there exists a unique solution passing through P and P' at these times, respectively.

(b) The path segment joining P and P' is composed solely of points of K.

3. We shall determine the circumstances under which a fixed set is ballistically closed with respect to certain second-order, linear differential systems in a Hilbert space and for this purpose certain preliminary matters will be taken up.

The discussion will be confined to the case of a real Hilbert space. The length of a vector x will be written $\|x\|$. The inner product of two vectors x and y will be written (x,y). All limits are to be taken in the sense of the strong topology, *i.e.*,

$$\lim_{n \to \infty} x_n = x$$

is to mean

$$\lim_{n \to \infty} \| x - x_n \| = 0 \quad .$$

In particular, a vector function $x(t)$ is (strongly) continuous if the sequence

$$x(t_1) \; , \; x(t_2) \; , \; \ldots \; , \; x(t_n) \; , \; \ldots$$

converges strongly to $x(t)$ whenever the sequence of argument values converges to t .

The <u>derivative</u> $\frac{dx}{dt}$ of a vector function is defined to be

$$\lim_{h \to 0} \frac{x(t+h) - x(t)}{h}$$

providing this (strong) limit exists. A vector function is <u>analytic</u> at a point t_o if its values in a suitable neighborhood of t_o may be represented by a power series in $(t-t_o)$ with vector coefficients, the series being convergent in the strong sense. It follows easily that such functions have analytic derivatives.

Unless otherwise stated, a vector x will be assumed to have its initial point at the origin and its terminal point will often be spoken of as "the point x ."

We shall be concerned with real, bounded, linear transformations. If A be such a transformation, then there is a least positive number k such that, for every vector x, we have

$$\|Ax\| \leqq k \|x\| \quad .$$

This number k is called the <u>bound</u> of A and we will write

$$\|A\| = k \quad .$$

The iterates of A will, as is usual, be written

$$A^1 = A \ , \ A^2 = AA \ , \ \ldots \quad .$$

If the vector equation

$$Ax = y$$

has a unique solution for each choice of y, and if the solution x is always such that, for a suitable fixed positive number k',

$$\|x\| \leq k' \|y\| \quad ,$$

then A has a bounded inverse denoted by A^{-1} .

4. Let

$$p(z) = a_o + a_1 z + a_2 z^2 + \dots + a_n z^n$$

$$= a_o(1+b_1 z)(1+b_2 z) \quad \dots \quad (1+b_n z)$$

be a polynomial which is real-valued for real z and for which all b_p are non-vanishing and real. The transformation

$$p(A) = a_o I + a_1 A + a_2 A^2 + \dots + a_n A^n$$

$$= a_o(I+b_1 A)(I+b_2 A) \dots (I+b_n A)$$

is then real, linear, and bounded so long as A is such. In fact,

$$\|p(A)\| \leq |a_o| + |a_1| k + |a_2| k^2 + \dots + |a_n| k^n \quad .$$

A propos of the second representation of $p(A)$, it is clear that $(p(A))^{-1}$ exists as a bounded transformation so long as

$$I + sA$$

has a bounded inverse when

$$s = b_1 , b_2 , \ldots , b_n .$$

In the usual terminology that set of numbers m for which

$$A - mI$$

has a bounded inverse is called the <u>resolvent set</u> of A and so $(p(A))^{-1}$ exists as a bounded transformation so long as the numbers

$$-\frac{1}{b_1} , -\frac{1}{b_2} , \ldots , -\frac{1}{b_n}$$

lie in the resolvent set of A .

From polynomials of transformations we pass to the consideration of entire functions of transformations. An infinite sequence of transformations

$$A_1 , A_2 , \ldots , A_n , \ldots$$

converges strongly whenever, for arbitrary m ,

$$\| A_{n+m} - A_n \|$$

can be made as small as we please by choosing n sufficiently large. Corresponding to such a sequence, there is a limit transformation A since the sequence of images of any vector x converges strongly. Also, A is bounded--more exactly,

$$\| A \| \leq \text{limes superior } \| A_n \| .$$

Suppose $f(z)$ is an entire function defined by

$$a_o + a_1 z + a_2 z^2 + \ldots + a_n z^n + \ldots ;$$

then

$$f(A) = a_o I + a_1 A + a_2 A^2 + \ldots + a_n A^n + \ldots$$

defines a bounded linear transformation which is real for A real so long as $f(z)$ is real for real z. Pursuing the matter further, let $f(z)$ have as its infinite product representation

$$a_o(1+b_1 z)(1+b_2 z) \ldots (1+b_n z) \ldots$$

where, for simplicity it has been supposed that the Weierstrass convergence factor is constant. The product is absolutely convergent by supposition and so

$$|b_1| + |b_2| + \dots + |b_p| + \dots$$

is a convergent sum. Set

$$B_n = a_o \prod_{p=1}^{n} (I+b_pA) \quad ;$$

then

$$\|B_{n+m} - B_n\| = \|B_n(A\sum_{n+1}^{m} b_p + A^2 \sum_{n+1}^{m}{}' b_p b_q + \dots$$

$$\dots + A^m \prod_{n+1}^{m} b_p)\| \quad ,$$

where the primed sign of summation means that those products having terms of equal index are to be omitted from the sum. Now

$$\|B_n\| \leqq |a_o| \prod_{p=1}^{n} (1+|b_p|k) <$$

$$< |a_o| \prod_{p=1}^{\infty} (1+|b_p|k) = k' < \infty$$

and for n sufficiently large

$$k\left|\sum_{n+1}^{m} b_p\right| \leqq \frac{h}{k'+h}$$

for all m regardless of how small a positive number h be taken. Further,

$$k^2\left|\sum_{n+1}^{m}{}' b_p b_q\right| < \left(\frac{h}{k'+h}\right)^2 ,$$

$$\cdots\cdots\cdots\cdots\cdots$$

$$k^m\left|\prod_{n+1}^{m} b_p\right| < \left(\frac{h}{k'+h}\right)^m .$$

Taking the sum of these bounds to infinity for good measure shows that

$$\left\| A\sum_{n+1}^{m} b_p + A^2 \sum_{n+1}^{m}{}' b_p b_q + \cdots + A^m \prod_{n+1}^{m} b_p \right\| < \frac{h}{k'} .$$

These results combine to prove that the sequence of transformations generated by the partial products is uniformly bounded and convergent, which allows us to write

$$f(A) = a_o \prod_{p=1}^{\infty} (I+b_p A)$$

and this is a bounded, linear transformation. It is clear from a comparison of the coefficients of corresponding iterates of A that the definitions of $f(A)$ by a power series and by an infinite product yield the same transformation.

As in the case of polynomials of transformations, the product representation has the advantage of showing whether or not $f(A)$ has a bounded inverse. Let us suppose that the numbers

$$-\frac{1}{b_1}, \; -\frac{1}{b_2}, \; \ldots, \; -\frac{1}{b_n}, \; \ldots$$

lie in the resolvent set of A. It will now be shown that $(f(A))^{-1}$ exists. Consider the sequence of transformations generated by the products

$$C_n = \frac{1}{a_o} \prod_{p=1}^{n} (I+b_pA)^{-1} \quad .$$

We have

$$C_nf(A) = f(A)C_n = \prod_{p=n+1}^{\infty} (I+b_pA)$$

and so

$$f(A)(C_{n+m}-C_n) = (C_{n+m}-C_n)f(A) =$$

$$= \left(\prod_{n+m+1}^{\infty} - \prod_{n+1}^{\infty}\right)(I+b_pA) \quad ,$$

which shows that the sequences

$$C_1f(A) \quad , \quad C_2f(A) \quad , \ldots$$

$$f(A)C_1 \quad , \quad f(A)C_2 \quad , \ldots$$

have a common limit

$$f(A)C = Cf(A) \quad .$$

Since

$$B_nC_n = I \quad ,$$

it follows that

$$C = \frac{1}{a_o}\prod_{p=1}^{\infty}(I+b_pA)^{-1} = (f(A))^{-1} \quad .$$

Moreover, $(f(A))^{-1}$ is bounded.

Finally, it is clear that if one of numbers $-\frac{1}{b_p}$ does not lie in the resolvent set of A, the order of factors in $f(A)$ being immaterial, $f(A)$ does not have a bounded inverse applicable to every vector of the space.

In summary:

__Lemma 1.__ __If__

$$\sum_{p=1}^{\infty} |b_p|$$

__converges and if the infinite product__

$$a_o \prod_{p=1}^{\infty} (1+b_p z)$$

__(which is absolutely and uniformly convergent for all finite__ z __) assumes real values for real values of__ z __,__ __then for any real, bounded, linear transformation__ A __, the product__

$$f(A) = a_o \prod_{p=1}^{\infty} (I+b_p A)$$

__is a real, bounded linear transformation. Further, it is necessary and sufficient that the numbers__

$$-\frac{1}{b_1}\ ,\ -\frac{1}{b_2}\ ,\ \dots\ ,\ -\frac{1}{b_n}\ ,\ \dots$$

<u>lie in the resolvent set of</u> A <u>in order that</u> $(f(A))^{-1}$ <u>exist as a bounded transformation applicable to all vectors.</u> <u>If this be the case, then</u>

$$(f(A))^{-1} = \frac{1}{a_o}\prod_{p=1}^{\infty}(I+b_pA)^{-1} \quad .$$

<u>The order of factors in these products is immaterial.</u>

CHAPTER II

1. We turn now to vector differential equations in a Hilbert space: Let B be a real, bounded, linear transformation not dependent upon t and consider the equation

$$\frac{dx}{dt} = Bx \quad .$$

There is a unique solution to every Cauchy problem for such an equation, <u>i.e.</u>, a solution which assumes a preassigned value x_o at a preassigned value of t ,

say $t = 0$.[2] To begin with, if there is a solution $x(t)$, it must be infinitely differentiable, the successive derivatives being given by the formula

$$\frac{d^p x}{dt^p} = B \frac{d^{p-1} x}{dt^{p-1}} \quad .$$

Moreover, this relation shows that

$$\left\| \frac{d^p x}{dt^p} \right\| \leqq k^p \| x \| \quad ,$$

where

$$k = \| B \| \quad .$$

The series

$$x = (I + Bt + \frac{B^2 t^2}{2!} + \ldots + \frac{B^p t^p}{p!} + \ldots) x_o$$

clearly converges for all t and satisfies our differential equation, as is seen by direct differentiation.

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[2] The problem (with B dependent upon t) was first considered by H. von Koch, W. L. Hart, and others. In particular, the paper of A. Wintner: "Zur Theorie der unendlichen Differentialsysteme," Mathematische Annalen, vol. 95, 1926, pp. 544-556, is most complete.

Since it assumes the desired Cauchy data at $t = 0$, it is a solution to the problem.

Because the system of differential equations is linear and homogeneous, the question of uniqueness is immediately reducible to showing that the only solution which vanishes at $t = 0$ is that which vanishes everywhere. In any case, the solution is as differentiable as we please and, applying Taylor's theorem, we may write

$$x(t) = \frac{1}{p!} \int_0^t (t-t')^p \frac{d^{p+1}}{dt^{p+1}} x(t')dt'$$

(the vector coefficients of the preceding terms vanishing in consequence of the formula for determining the successive derivatives at $t = 0$). For a fixed value of t, say t_0, let M be the maximum of the length of $x(t)$ over the t-interval from zero to t_0. We have

$$\| x(t) \| \leqq \frac{k^{p+1} |t_0|^{p+1}}{(p+1)!} M .$$

From this we may conclude that, by taking p sufficiently large $\| x \|$ may be made arbitrarily small. That is, $x(t)$ vanishes identically, as was to be shown.

Finally, it may be shown from the series representation of the solution to the Cauchy problem, that the solution is

entire of order one, in the sense that any component of x is such a function of t.

2. Next, we treat the Cauchy problem for the equation

$$\text{(E)} \qquad \frac{d^2x}{dt^2} = Ax \quad ,$$

where it is supposed that A is a real, bounded, linear transformation which does not depend on t. The quantity k will now denote the bound of A. The preassigned Cauchy data are taken at $t = 0$, where it is supposed that

$$x = x_o \quad ,$$

$$\frac{dx}{dt} = w_o \quad .$$

If H be the Hilbert space in which our problem is set, we proceed to formulate the problem in the Cartesian product space

$$H_1 = H \times H \quad .$$

To be clear, suppose that the components of x and $\frac{dx}{dt}$ with respect to some complete orthonormal system are

$$x_1 , x_2 , \ldots , x_n , \ldots$$

and

$$\frac{dx_1}{dt} , \frac{dx_2}{dt} , \ldots , \frac{dx_n}{dt} , \ldots .$$

With this pair of vectors in H we associate a single vector z in H_1 according to the rule

$$z_1 = x_1 , z_3 = x_2 , \ldots , z_{2p-1} = x_p , \ldots ,$$

$$z_2 = \frac{dx_1}{dt} , z_4 = \frac{dx_2}{dt} , \ldots , z_{2p} = \frac{dx_p}{dt} , \ldots .$$

With the transformation A in H we associate a transformation B in H_1 which is such that in matrix notation

$$B = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & \cdots \\ a_{11} & 0 & a_{12} & 0 & a_{13} & 0 & \cdots \\ 0 & 0 & 0 & 1 & 0 & 0 & \cdots \\ a_{21} & 0 & a_{22} & 0 & a_{23} & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 1 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix} ,$$

where the quantities a_{pq} form the matrix of A in H. Clearly $\|B\|$ = maximum of 1 and k.

Equation (E) reads

$$\frac{dz}{dt} = Bz \quad ,$$

with Cauchy data at $t = 0$

$$z = z_o \quad .$$

From the previous results we have:

There is a unique solution to the Cauchy problem for equation (E), which is an analytic vector function and entire of order one.

This solution may be specifically exhibited in the following form:

Let

$$A_1(t) = I + \frac{At^2}{2!} + \frac{A^2t^4}{4!} + \dots + \frac{A^pt^{2p}}{(2p)!} + \dots \quad ,$$

$$A_2(t) = It + \frac{At^3}{3!} + \frac{A^2t^5}{5!} + \dots + \frac{A^pt^{2p+1}}{(2p+1)!} + \dots \quad ;$$

then

$$x(t) = A_1 x_o + A_2 w_o \quad .$$

3. Finally, let us consider the boundary-value problem for equation (E). We seek a solution $x(t)$ such that

$$x(0) = x_o \quad ,$$

$$x(t_1) = x_1 \quad ,$$

x_o and x_1 being preassigned vectors in K. If a solution exists, we must be able to determine a w_o appropriate to the Cauchy problem from a knowledge of x_o and x_1. Moreover, if the solution is to be unique, so also must be this determination of w_o. This is to say that, in order for the boundary-value problem to have a solution which is unique, the equation

$$A_2(t_1) w_o = x_1 - A_1(t_1) x_o$$

must have a solution for w_o which is unique for any choice of x_o and x_1 in K.

It is clear then that the foregoing equation must have a unique solution throughout a spherical neighborhood of some point. For if we fix x_0 to be a point of K and allow x_1 to move through a spherical neighborhood of some interior point of K, the point

$$x_1 - A_1(t_1)x_0$$

will sweep out a translated spherical neighborhood, throughout which the equation must be uniquely solvable.

If this spherical neighborhood should contain the origin as an interior point, it follows immediately that the equation must be uniquely solvable throughout the whole space.

But this is equally true should the spherical neighborhood not contain the origin as an interior point. For, let c be the conical region, one sheet of whose boundary subtends the spherical neighborhood. We may suppose c to be closed by taking, if necessary, a closed spherical neighborhood interior to the original one. From the linearity of $A_2(t_1)$, it follows that the equation must be uniquely solvable throughout c. Now let y be any vector not in c. Let y' be a generator of c such that the plane p determined by y and y' is not tangent to the cone c. This plane p will meet c in a second generator y''. Then we may write

$$y = y_1 + y_2 \quad ,$$

where y_1 and y_2 are vectors in the directions of y' and y''. Let x_1 and x_2 satisfy

$$A_2(t_1)x_1 = y_1 \quad ,$$

$$A_2(t_1)x_2 = y_2 \quad ;$$

these solutions exist on account of the position of y_1 and y_2. The vector

$$x = x_1 + x_2$$

satisfies

$$A_2(t_1)x = y \quad .$$

This solution is unique. Suppose it were not and let $\bar{x}$ be a second solution. Then

$$A_2(t_1)(x-\bar{x}) = 0 \quad .$$

Since the origin is in the closed conical set c, in which the equation has a unique solution, we must have

$$x = \bar{x} \quad .$$

With regard to $A_2(t_1)$ we know then that it is bounded and the equation

$$A_2(t_1)x = y$$

has a unique solution for all vectors y. Banach has proven that under these circumstances $A_2(t_1)$ has a bounded inverse. [3]

Set

$$A_2(t) = tA_3(t)$$

where

$$A_3(t) = I + \frac{At^2}{3!} + \dots + \frac{A^p t^{2p}}{(2p+1)!} + \dots \quad .$$

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[3] See E. Hille, Functional Analysis and Semi-Groups, American Mathematical Society, New York, 1948, p. 29, corollary to Theorem 2.13.3.

Now

$$\frac{\sinh z^{1/2}t}{z^{1/2}t} = 1 + \frac{zt^2}{3!} + \dots + \frac{z^p t^{2p}}{(2p+1)!} + \dots =$$

$$= \prod_{p=1}^{\infty}\left(1 + \frac{zt^2}{\pi^2 p^2}\right) ,$$

which series and product are absolutely and uniformly convergent for all z and so

$$A_3(t) = \prod_{p=1}^{\infty}\left(I + A\frac{t^2}{\pi^2 p^2}\right) .$$

In virtue of Lemma 1 at the end of section 1 of the first chapter, we have:

Lemma 2. In order that the boundary-value problem for equation (E) have a unique solution for any pair of points in K, it is necessary and sufficient that the numbers

$$-\frac{\pi^2}{t_1^2} , \quad -\frac{4\pi^2}{t_1^2} , \dots , \quad -\frac{p^2\pi^2}{t_1^2} , \dots$$

lie in the resolvent set of A. The solution depends continuously on the given data. If, therefore, a unique solution exists for all positive t, the whole negative real axis must belong to the resolvent set.

CHAPTER III

1. In this chapter we will assume that A is a real, bounded, linear transformation, independent of t, whose resolvent set includes the entire negative real axis (though not necessarily the origin). Then, according to Lemma 2, we know that all boundary-value problems are solvable for the vector equation

$$\text{(E)} \qquad \frac{d^2x}{dt^2} = Ax \quad .$$

The solutions depend continuously upon the boundary data. We propose to determine the necessary and sufficient conditions upon A which insure that a specified set K be ballistically closed with respect to equation (E). With regard to K, it will be assumed throughout that:

(a) K <u>contains an interior point</u>, <u>i.e.</u>, there is a point x such that for suitably small positive h, those points y which satisfy

$$\| x-y \| < h$$

are in K.

(b) K <u>is closed in the sense of the strong topology</u>, <u>i.e.</u>, if

$$x_1 , x_2 , \ldots , x_n , \ldots$$

be a strongly convergent sequence of points lying in K, then the limit point of the sequence lies in K.

If a point x of K has the property that, for any positive h, there are vectors y not in K which satisfy

$$\| x-y \| < h \quad ,$$

then x is a <u>boundary point</u> of K.

The entire space is clearly ballistically closed with respect to equation (E). We shall disregard this trivial case and so the closed sets K of which we speak will always have some boundary points.

2. We first show that a ballistically closed set K is convex, that is, the straight line segment joining any two points of K lies wholly in K. Consider the solutions to equation (E) passing through x_o at $t = -t_o$ and through y_o at $t = +t_o$. If K is a ballistically closed

set and x_o and y_o are points of K, then by allowing t_o to vary we obtain a family of path segments lying in K. The same family of curves can be generated as solutions to

$$\frac{d^2x}{dt^2} = t_o{}^2 Ax \ ,$$

$$(E') \qquad x(-1) = x_o \ ,$$

$$x(+1) = y_o \ .$$

Allow t_o to approach zero. The path segment correspondingly approaches the straight line segment joining x_o to y_o. Since K is closed, it must contain this line segment. We have then

Lemma 3. If K is ballistically closed with respect to equation (E), then K is convex.

At this point, we may remark that the assumption that K contains an interior point is not a real restriction. For, if K does not contain an interior point, from its convexity we see that it must lie in some linear subspace. Any solution lying in this set K must have, as its derivatives, vectors which lie in this linear subspace.

Consequently, it follows that A and its iterates transform this subspace into itself. *E.g.*, if the subspace be those vectors x satisfying

$$(x,u) = 0 \quad ,$$

we have for any solution lying in K

$$\left(\frac{d^2x}{dt^2},u\right) = (Ax,u) = 0 \quad .$$

Thus the last equation holds for all x in K. But assuming in this example that K has interior points with respect to the linear subspace, it follows that

$$(Ax,u) = 0$$

for all x in the linear subspace.

So, leaving aside the trivial case in which K consists of but a single point, we see that K must lie in some linear subspace with respect to which it does contain an interior point. In this case the problem may be reformulated in this subspace.

We shall subsequently make use of the following material from the theory of convex sets. Let u be a fixed vector, c a fixed constant, and denote by $L(x)$ the linear form $(u,x) - c$. The set of vectors which cause this linear form to vanish form a hyperplane L which divides the space into two parts, namely: the set of vectors y for which

$$L(y) > 0 \quad ,$$

which is called the <u>positive half-space</u>; and the set of vectors z for which

$$L(z) \leqq 0 \quad ,$$

which is called the <u>non-positive half-space</u>. Observe that this convention assigns an <u>orientation</u> to the hyperplane, that is, the hyperplane

$$-L(x) = 0$$

yields reversed positive and non-positive half-spaces.

Such an oriented hyperplane is a <u>supporting hyperplane</u> of a closed convex set K providing that:

(a) If y is in the positive half-space of L, then y is not in K.

(b) There is a point x of K which is in L.

At each boundary point x of K there is at least one supporting hyperplane of K. [4]

3. Suppose now that K is ballistically closed and hence closed and convex. Let x_o be a boundary point of K. We shall show that the vector Ax_o, when drawn from x_o, lies on the non-negative side of every supporting hyperplane to K at x_o. Suppose the contrary to be the case. We set the Cauchy problem with data

$$x(0) = x_o ,$$

$$\left.\frac{dx}{dt}\right|_{t=0} = 0 .$$

For positive t_o sufficiently small, the point $x(t_o)$ of the solution is an interior point of K if the supporting hyperplane is unique, since

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[4] T. Bonnesen and W. Fenchel, Theorie der Konvexen Körper, Ergebnisse der Math., vol. 3, no. 1, 1934, pp. 5-6. While the proof is for a finite dimensional space, a modified proof for a Hilbert space is easily obtained.

$$x(t_o) = x_o + \left(\frac{d^2x}{dt^2}\right)\Bigg|_{t=0} t^2/2! + \dots =$$

$$= x_o + Ax_o t^2/2! + \dots \quad .$$

We now alter the second item of Cauchy data by taking

$$\frac{dx}{dt}\Bigg|_{t=0} = w_o$$

to be a vector which, when drawn from x_o , goes outside K . If $\|w_o\|$ be taken sufficiently small and $z(t)$ be the solution to this problem, then we have

$$\|z(t_o) - x(t_o)\| < h$$

where h may be taken arbitrarily small, but positive. Thus $z(t_o)$ is in K . But in this case K could not be ballistically closed, for the solution joining x_o and $z(t_o)$, which are points of K , would pass outside K in virtue of the choice of direction of w_o .

But the argument is equally valid if the supporting hyperplane at x_o is not unique. For we shall show that if Ax_o lies on the negative side of <u>some</u> supporting hyperplane to K at x_o , then a neighboring point x_o' can be

found such that $\frac{t^2}{2!} Ax_o'$, when drawn from x_o' , terminates in the interior of K for suitably small t . The construction is as follows: From x_o lay off a segment m which lies in the interior of K (with the exception of x_o , of course). Then, from a point on m , which is at a distance d from x_o , draw a ray in the direction of $-Ax_o$. Such a ray will meet the boundary of K at a point $x_o(d)$. Consider the vector $Ax_o(d)$, drawn from $x_o(d)$. If d be chosen to be sufficiently small, then the direction of $Ax_o(d)$ can be made to be as close as we please to the direction of Ax_o . Now, from the convexity of K and the construction whereby $x_o(d)$ was located, we see that Ax_o , when drawn from $x_o(d)$, goes into the interior of K . So it follows that if d be taken small enough, say equal to d_o , then any suitably short segment, laid off from $x_o(d_o)$ in the direction of $Ax_o(d_o)$ must lie wholly in the interior of K . That is, taking x_o' to be $x_o(d_o)$ and t to be small, the vector $\frac{t^2}{2!} Ax_o'$ when drawn from x_o' terminates in the interior of K . The argument of the preceding paragraph may then be reapplied.

A transformation A will be called <u>boundary-extending for</u> K when the following condition is satisfied for every boundary point x of K : The vector Ax , when drawn from x, lies in the non-negative half-space of every supporting hyperplane to K at x .

Using this terminology, we have proven:

Lemma 4. *In order that* K *be ballistically closed with respect to equation* (E), *it is necessary that* A *be boundary-extending for* K.

We next prove:

Lemma 5. *If* K *is ballistically closed with respect to equation* (E), *then there is a point* $\bar{x}$ *of* K *for which*

$$A\bar{x} = 0 \quad .$$

The translation

$$x - \bar{x} = y$$

changes equation (E) *into*

$$\frac{d^2y}{dt^2} = Ay$$

and the translated set contains the origin (possibly as a boundary point).

Let x_o be a fixed interior point of K. As the point x moves through the points of K, the point

$$y(r) = x - x_o + rAx$$

(where r is a real scalar parameter) moves through a set $K'(r)$. We repeat for emphasis that $I + rA$ is assumed to be non-singular for $r \geqq 0$. Let S and $S'(r)$ be the corresponding boundary hypersurfaces of these sets. When r equals zero, then $K'(0)$ is just the set K translated in such a way that the interior point x_o of K is sent into the origin. Hence the origin is an interior point of $K'(0)$. It is clear that for r sufficiently small, but positive, this is equally true of $K'(r)$. For $r > 0$ set

$$z(r) = y(r)/r \quad .$$

As $y(r)$ traces out $K'(r)$, $z(r)$ traces out a set $K''(r)$ which, again for sufficiently small positive r, contains the origin as an interior point. Let $S''(r)$ be the boundary surface of $K''(r)$. Suppose that K is ballistically closed with respect to equation (E). A is boundary-extending for K by the previous lemma. It is clear, then, that the surface $S'(r)$ and hence also $S''(r)$ cannot pass through the origin when r is positive. Thus, as we allow r to increase, the origin must remain an interior point of $K''(r)$. Consider the limit set

obtained from $K''(r)$ as r increases without bound. This set is described by the vector Ax drawn from the origin as x moves through the points of K. Since this limit set must contain the origin either as an interior point or possibly as a boundary point, we see that for some vector of K, say $\bar{x}$, we have

$$A\bar{x} = 0 \quad .$$

The rest of the lemma is then obvious.

4. We summarize the conditions upon K. They are:

(a) K _contains an interior point._

(b) K _is closed in the strong topology._

(c) K _is convex._

(d) K _contains the origin._ [5]

Let us consider, as preparation for further considerations, the effect of a bounded, linear transformation B upon such a set K. In virtue of the linearity of B, the image set K' satisfies conditions (c) and (d). If, further, B has a bounded inverse, then conditions (a) and (b) are also plainly satisfied by K'. Under such circumstances,

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[5] _I.e._, we shall always suppose the translation described in Lemma 5 to have been performed.

there are two possibilities of special interest: namely, that in which K includes K' and that in which K' includes K. In the first case we shall say B shrinks K and in the second case that B expands K.

Observe that we have so far found as necessary conditions on A:

(a) The resolvent set of A includes the entire negative real axis.

(b) A is boundary-extending for K.

The following lemma gives a single alternative condition.

Lemma 6. Conditions (a) and (b) are necessary and sufficient to insure that the transformation

$$I + s^2 A$$

expands K for each choice of the real parameter s.

We shall first show the conditions to be necessary.

Condition (a) follows immediately from the fact that an expanding transformation must have a bounded inverse.

Regarding condition (b), we remark that, under such a linear transformation, boundary points of a set K are transformed into boundary points of the image set. Let x be a boundary point of K and suppose Ax is on the

negative side of some supporting plane of K at x. First, suppose the supporting hyperplane at x to be unique. Then for s^2 suitably small, s^2Ax, when drawn from x, i.e., $(I + s^2A)x$, must lie in the interior of K contrary to the fact that it must be a boundary point of a set which includes K. If the supporting hyperplane at x is not unique, the construction used in the proof of Lemma 4 may be reapplied. Thus the conditions are necessary.

The conditions are also sufficient.

To see this, we first observe that for every choice of s^2, if x be a boundary point of K, then

$$(I + s^2A)x$$

cannot be an interior point of K.

We next propose to show that for s^2 sufficiently small there are interior points of K which are, at the same time, interior points of $K(s^2)$, the image of K under the transformation $I + s^2A$.

Let y be an interior point of K. For sufficiently small values of h, all points z satisfying

$$\| z - y \| < h$$

are also within K. Let $y(s^2)$ denote the image of y under $I + s^2A$. We have

$$\| y(s^2) - y(0) \| \leqq s^2 k \| y \| \quad ,$$

and so if s^2 be chosen so that

$$s^2 < \frac{h}{k \| y \|} \quad ,$$

then $y(s^2)$ certainly lies within K.

Further, for s^2 less than a suitably chosen h', the transformation $(I + s^2A)^{-1}$ exists since it may be expanded in a convergent geometrical series in s^2A. We need only take h' to satisfy

$$h' < \frac{1}{\| A \|^{1/2}} \quad .$$

Now let s^2 be less than either h or h'. For such values of s^2, it follows that $K(s^2)$ includes K. For, it if were otherwise, in virtue of the non-singularity of $I + s^2A$, we could find a point $z(s^2)$ not in $K(s^2)$, but interior to K.

Then, because of the convexity of K, there must be an interior point of K

$$u(s^2) = (1-r)z(s^2) + ry(s^2)$$

with

$$0 < r < 1$$

(where $y(s^2)$ is an interior point common to K and $K(s^2)$) such that $u(s^2)$ is on the boundary of $K(s^2)$. Since the boundary points of $K(s^2)$ cannot be interior points of K , the contradiction proves the point.

So far we have proven that, for small s^2 , the transformation $I + s^2A$ expands K . From this result and assumption (a), we can show this to be the case for all real s^2 . We have seen that boundary points of K are transformed into points either outside or on the boundary of K no matter how large s^2 be taken. We shall demonstrate that for each choice of s^2 there is an interior point y in K such that its image $y(s^2)$ is also an interior point of K . When this has been done, we need only reapply the argument of the preceding paragraph to see that the lemma is true.

Suppose, to the contrary, that for some s^2 , say s_o^2 , we have the following situation: y is an interior point of K and its image $y(s_o^2)$ lies outside or on the

boundary of K. Then, in virtue of the non-degeneracy of the transformation, there must be a point z, lying outside K, whose image $z(s_o^2)$ is an interior point of K. The course of $z(s^2)$ from $s^2 = 0$ to $s^2 = s_o^2$ is a continuous one. Hence, for some intermediate value of s^2, $z(s^2)$ must be a boundary point. But such boundary points are always transformed into points outside or on the boundary of K for all subsequent values of s^2. This precludes $z(s_o^2)$ being an interior point of K. The lemma is thus completely proven.

The next two lemmas furnish further alternatives to conditions (a) and (b) in the finite dimensional case.

Lemma 6'. *In the finite dimensional case, condition* (a) *may be replaced by*

$$\det(I + s^2A) > 0$$

for all real s.[6)]

This follows immediately from the fact that this determinant is a polynomial in s^2 which is equal to one when

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6) det B = determinant formed from matrix B.

s^2 vanishes. If the resolvent set includes the entire negative real axis, the polynomial has no real roots.

Lemma 6". *In the finite dimensional case, if the origin is an interior point of* K *, then condition* (b) *alone implies that*

$$I + s^2 A$$

expands K .

By Lemma 6', we need only prove that

$$\det(I + s^2 A) > 0$$

for all real s . It is certainly so for sufficiently small values of s^2 . Suppose then that for some s_0^2 we have

$$\det(I + s_0^2 A) = 0 \quad .$$

Then there must be a point x other than the origin such that

$$(I + s_0^2 A)x = 0 \quad .$$

Let y be that point of the boundary of K which lies on the ray from the origin through x. For a suitable positive scalar r^2, we must have

$$y = r^2 x \ .^{7)}$$

Whence

$$(I + s_o^2 A)y = r^2(I + s_o^2 A)x = 0 \quad .$$

From the boundary-extending property of A, we have a contradiction, for $(I + s_o^2 A)y$ must lie outside or on the boundary of K. This proves that

$$\det(I + s^2 A) > 0 \quad .$$

To return to the mainstream of our discussion, we have, in virtue of Lemma 6: <u>It is necessary, in order for</u> K <u>to be ballistically closed with respect to equation</u> (E), <u>that</u> $I + s^2 A$ <u>expands</u> K <u>for all real</u> s.

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7) The positive character of r^2 follows from the fact that there is a non-zero minimum distance from the origin to points of K.

5. This condition will now be shown to be sufficient.

As a preliminary matter, we shall exhibit a particular form of the general solution to the boundary-value problem. Since equation (E) is translation-invariant with respect to the time, we assume without loss of generality that the end point values of t are

$$-t_o \quad \text{and} \quad t_o \quad ,$$

where t_o is assumed positive but otherwise arbitrary. Set

$$A_2(s) = s \prod_{p=1}^{\infty} \left(I + \frac{As^2}{\pi^2 p^2}\right) \quad .$$

The general solution of the boundary-value problem is

$$x = A_2(t_o - t)(A_2(2t_o))^{-1} x_o +$$

$$+ A_2(t_o + t)(A_2(2t_o))^{-1} y_o \quad ,$$

where x_o and y_o are the prescribed values of x assumed at $-t_o$ and t_o, respectively. (Heuristically speaking, if (E) were a scalar equation, then the solution would be:

$$x = \frac{\sinh A^{1/2}(t_o - t)}{\sinh 2A^{1/2}\ t_o} x_o +$$

$$+ \frac{\sinh A^{1/2}(t_o + t)}{\sinh 2A^{1/2}\ t_o} y_o$$

and the above solution is defined via the infinite product representation of these functions.) By assumption the numbers

$$-\frac{\pi^2}{4t_o^2}, \quad -\frac{\pi^2}{t_o^2}, \quad -\frac{9\pi^2}{4t_o^2}, \ldots$$

are in the resolvent set of A and so $(A_2(2t_o))^{-1}$ exists. Then, in virtue of Lemma 1 in Chapter I, we may write

$$A_2(t_o-t)(A_2(2t_o))^{-1} =$$

$$= \left(\frac{t_o-t}{2t_o}\right) \prod_{p=1}^{\infty} \left(I + A\frac{(t_o-t)^2}{\pi^2 p^2}\right)\left(I + A\frac{(2t_o)^2}{\pi^2 p^2}\right)^{-1} .$$

Now we suppose that

$$I + s^2 A$$

expands K to $K(s^2)$ for each choice of s^2 . Then it follows that

$$(I + s^2A)^{-1}$$

shrinks $K(s^2)$ to K. Consequently this latter transformation shrinks K itself.

Now let x be a vector in K and set

$$y = (I + s^2A)^{-1}x \quad .$$

The vector y is also in K. Let

$$z = (I + As'^2)y \quad ,$$

where

$$0 \leqq s'^2 \leqq s^2 \quad ;$$

then

$$z = \frac{(s^2-s'^2)y + s'^2x}{s^2} \quad .$$

Since z is thus written in the form

$$rx + (1-r)y$$

with

$$0 \leqq r \leqq 1 \quad ,$$

it lies in K. In case

$$s^2 = \frac{(2t_o)^2}{\pi^2 p^2} \quad ,$$

$$s'^2 = \frac{(t_o - t)^2}{\pi^2 p^2} \quad , \quad p = 1,2,\ldots \quad ,$$

each product

$$(I + A\frac{(t_o-t)^2}{\pi^2 p^2})(I + A\frac{(2t_o)^2}{\pi^2 p^2})^{-1}$$

shrinks K. If we set

$$u(t) = \prod_{p=1}^{\infty}(I + A\frac{(t_o-t)^2}{\pi^2 p^2})(I + A\frac{(2t_o)^2}{\pi^2 p^2})^{-1} x_o \quad ;$$

$$v(t) = \prod_{p=1}^{\infty}(I + A\frac{(t_o+t)^2}{\pi^2 p^2})(I + A\frac{(2t_o)^2}{\pi^2 p^2})^{-1} y_o \quad ;$$

it is clear that when x_o and y_o are points of K, so are $u(t)$, $v(t)$ for the t-values considered between $-t_o$ and t_o. But then the general solution of the boundary-value problem is

$$x(t) = \left(\frac{t_o - t}{2t_o}\right)u(t) + \left(\frac{t_o + t}{2t_o}\right)v(t) \quad .$$

Notice that the coefficients of u and v lie between zero and one and that their sum is one, for the values of t considered.

In virtue of the convexity of K, $x(t)$ lies in K, as was to be shown.

In summary:

Theorem I. In order that a closed set K, containing an interior point, be ballistically closed with respect to the equation

$$\text{(E)} \qquad \frac{d^2x}{dt^2} = Ax \quad ,$$

it is both necessary and sufficient that:

(a) K is convex.

(b) Either K contains the origin or can be made to do so after a suitable translation which does not alter the problem.

(c) The transformation $I + s^2A$ expands K for each choice of the real parameter s. [8]

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[8] In (c) it is assumed that the translation mentioned in (b) has already been performed when necessary.

CHAPTER IV

1. In this chapter equation (E) is to be thought of as a finite system of equations in an equal number of unknowns. The vector x has components

$$x_1 , x_2 , \dots , x_n$$

with respect to a fixed Cartesian coördinate system. The transformation A has as its matrix

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} .$$

By $\det A$ is meant the determinant of A . By A' is meant the transpose of A , that is, the matrix whose elements are

$$a'_{pq} = a_{qp} .$$

All the quantities considered are real unless otherwise stated.

2. We shall now determine the necessary and sufficient conditions upon A which insure that for all real s the transformation

$$I + s^2 A$$

expands the set of all vectors whose components are all non-negative, *i.e.*, the first 2^n-tant. For brevity we call this set Q_n.

Suppose first that the transformation is known to expand Q_n; then its inverse shrinks Q_n. For s^2 sufficiently small

$$B = (I + s^2 A)^{-1} = I - s^2 A + s^4 A^2 - \dots \quad .$$

In order that this transformation shrinks Q_n, it is necessary that $b_{pq} \geqq 0$ for all choices of p and q. To see this, we need only to consider the images of the unit base vectors under B. These form the columns of B and must have all non-negative components. If $p \neq q$

$$b_{pq} = - s^2 a_{pq} + s^4 \sum_r a_{pr} a_{rq} + \dots \quad ,$$

from which we see that we must have

$$a_{pq} \leqq 0 \quad \text{for} \quad p \neq q \quad .$$

This condition is not only necessary but also sufficient in order that A extend the boundary of Q_n. To see this, we need only show that every boundary vector has at least one non-positive component. Suppose x be a boundary vector; for some p we must have

$$x_p = 0 \quad .$$

The p^{th} component of Ax is

$$a_{p1}x_1 + a_{p2}x_2 + \dots + a_{pp-1}x_{p-1} + a_{pp+1}x_{p+1} + \dots + a_{pn}x_n \quad .$$

Since

$$x_q \geqq 0 \quad , \quad q = 1,2,\dots,p-1 \; , \; p+1 \; , \; \dots \; , \; n \quad ,$$

it follows that this component is indeed non-positive.

By Lemma 5 we see that, <u>for</u> $I + s^2A$ <u>to expand</u> Q_n, <u>it is necessary and sufficient that</u>

(1) $a_{pq} \leqq 0$, $(p \neq q)$

(2) $\det (I + s^2A) > 0$,

<u>for all real</u> s .

We next obtain a further necessary condition. From

$$I = (I + s^2A)B$$

we have

$$1 = (1 + s^2 a_{pp})b_{pp} + s^2 \sum_{p \neq q} a_{pq} b_{qp}$$

or, since b_{qp} is non-negative for small s , and a_{pq} non-positive,

$$(1 + s^2 a_{pp})b_{pp} = 1 - s^2 \sum_{p \neq q} a_{pq} b_{qp} \geqq 1 \quad .$$

Now, because b_{pp} and $(1 + s^2 a_{pp})$ are continuous functions of s which are positive at $s = 0$, we have, for all s ,

(3) $b_{pp} > 0$ and $a_{pp} \geqq 0$.

We are now in a position to prove the following theorem which gives conditions on A which are free of the parameter s .

Theorem II. *In order that Q_n be ballistically closed with respect to equation (E), it is necessary and sufficient that:*

(a) *The elements of A which are not on the principal diagonal are non-positive.*

(b) *All principal minors of A (including the principal diagonal elements of A and the determinant of A) are non-negative.*

Condition (a), which is condition (1) is just the necessary and sufficient condition that A be boundary-extending for Q_n .

Condition (b) is sufficient to insure condition (2). For, let

$$a_p = \text{sum of all principal minors of order } p \text{ .}$$

In virtue of condition (b), these quantities are all non-negative. But then

$$\det(I + s^2A) = 1 + a_1s^2 + a_2s^4 + \ldots + a_ns^{2n}$$

is manifestly greater than zero for real s .

We shall finally show that the necessary conditions (1), (2), and (3) imply condition (b). The proof will be inductive.

The theorem is plainly true for $n = 1$.

For general n , suppose that Q_n is ballistically closed, that is, A satisfies conditions (1), (2), and (3). We take as the induction hypothesis that in an n-1 dimensional space when a transformation satisfies conditions (1), (2), and (3), it satisfies condition (b); i.e., a linear transformation which expands Q_{n-1} necessarily satisfies (b).

From condition (2) we have

$$\det(\frac{I}{s^2} + A) > 0$$

and so, allowing s to become large without bound, it follows that

$$\det A \geqq 0 \quad .$$

Thus it is necessary that the sole principal minor of order n is non-negative.

Next, we see that the principal minors of order $n-1$ in the matrix $I + s^2A$ are positive, because, by condition (3), the positive function b_{pp} is obtained from $I + s^2A$ by dividing one of its principal minors (the algebraic complement of $1 + s^2a_{pp}$) by the positive determinant $\det(I + s^2A)$.

Now consider the transformation whose matrix is obtained from $I + s^2A$ by striking out the p^{th} row and p^{th} column of this matrix. Its determinant is positive, by the preceding paragraph. Its non-diagonal elements are non-positive by condition (1). Hence it satisfies the conditions (1) and (2) which are sufficient to insure that it expands Q_{n-1} . But for such a transformation we have assumed as an induction hypothesis that it necessarily satisfies conditions (a) and (b). Thus, it follows that the principal minor of A gotten by striking out the p^{th} row and p^{th} column satisfies condition (b). This completes the proof that A necessarily satisfies condition (b) and so the theorem is completely demonstrated.

3. Next, we take up the case of the simplex having a vertex at the origin. First, suppose the $n+1$ vertices to be

$$\begin{array}{c} (0\ ,\ 0\ ,\ \ldots\ ,\ 0) \\ (1\ ,\ 0\ ,\ \ldots\ ,\ 0) \\ (0\ ,\ 1\ ,\ \ldots\ ,\ 0) \\ \cdots\cdots\cdots \\ (0\ ,\ 0\ ,\ \ldots\ ,\ 1)\ . \end{array}$$

The simplex (which we shall call S_n) is then bounded by the $n+1$ hyperplanes

$$\begin{array}{c} x_1 = 0 \\ x_2 = 0 \\ \cdots \\ x_n = 0 \end{array}$$

and

$$x_1 + x_2 + \ldots + x_n = 1 \quad .$$

It is clear that if $I + s^2A$ is to expand S_n, it must expand the first 2^n-tant. It only remains to specialize that class of transformations A described in the first section of this chapter in such a way that the image of the face arising from the last mentioned bounding hyperplane lies outside S_n. If e denote that vector whose components are all equal to one, then the last bounding hyperplane is

$$(x,e) = 1 \quad .$$

This is a supporting hyperplane to S_n. It is clear that the further necessary and sufficient condition sought is just that A be boundary-extending over this face also. Thus, we must have, for all x in this face,

$$(Ax,e) \geqq 0 \quad .$$

By choosing for x successively all those vertices of S_n, other than the origin, we are led to the conditions

$$\sum_p a_{pq} \geqq 0 \quad .$$

We may now treat the general simplex with one vertex at the origin. Suppose the remaining vertices to be points whose coördinates are the columns of a matrix B. In order that this be an n-dimensional simplex, it is necessary and sufficient that

$$\det B \neq 0 \quad .$$

If $I + s^2A$ expands this simplex, then

$$B^{-1}(I + s^2A)B = I + s^2(B^{-1}AB)$$

expands the simplex just previously discussed. In summary:

<u>Theorem III.</u> <u>In order that the n-dimensional simplex, having as its $n+1$ vertices the origin and those points whose coördinates form the columns of the non-singular matrix B, be ballistically closed with respect to equation (E), it is necessary and sufficient that the matrix</u>

$$C = B^{-1}AB$$

<u>have the following properties:</u>

(a) <u>The principal minors of C (including the elements of principal diagonal and the determinant of C) are non-negative.</u>

(b) <u>The elements of C not on the principal diagonal are non-positive.</u>

(c) <u>The sum of the elements of any column of C is non-positive.</u>

4. Now follows the case of the set S composed of points lying within or on the unit hypersphere centered at the origin. By Lemma 6', we need only show that A extends the boundary of S. If y be a point in the supporting hyperplane to this hypersphere at a point x, then y satisfies

$$(y,x) - 1 = 0 \quad .$$

Thus, in order that A extend the boundary of S, *i.e.*, in order that Ax when drawn from a boundary point x of S be on the non-negative side of this hyperplane, it is necessary and sufficient that

$$(Ax,x) \geqq 0$$

when

$$(x,x) = 1 \quad .$$

That is to say that the (symmetric) matrix $A + A'$ must be the coefficient matrix of a non-negative quadratic form.

The hyperellipsoidal case is equally simple, being affine-equivalent to the preceding. Suppose the hyper-ellipsoid to have the equation

$$(x,Bx) = 1 \quad ,$$

where B is the (symmetric) coefficient matrix of a positive definite quadratic form. Just as before, it is necessary and sufficient that

$$(Ax,Bx) = (x,A'Bx) \geqq 0$$

which is to say that the symmetric matrix $A'B + BA$ must be the coefficient matrix of a non-negative quadratic form.

We have

Theorem IV. *In order that the hyperellipsoidal body, centered at the origin and having as the equation of its bounding surface*

$$(x, Bx) = 1$$

(where B is the symmetric coefficient matrix of a positive definite quadratic form), be ballistically closed with respect to equation (E), it is necessary and sufficient that the symmetric matrix

$$A'B + BA$$

be the coefficient matrix of a non-negative quadratic form.

5. In conclusion, we give a simplification of the condition of Lemma 6' of the previous chapter for the case of a convex conical region C, having its vertex at the origin.[9] The conditions will be free of the parameter s. We state this as:

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9) C is to include only one sheet of the cone, of course.

<u>Theorem V.</u> <u>In order that C be ballistically closed with respect to equation (E), it is necessary and sufficient that:</u>

(a) <u>A extends the boundary of C.</u>

(b) <u>If a_p denotes the sum of the principal minors of order p, then the first non-vanishing number of the sequence</u>

$$a_n , a_{n-1} , \dots , a_1 , 1$$

<u>is positive.</u>

Upon referring to conditions (a) and (b) on page 40, and Lemma 6' on page 45, it is seen that we need only show that:

(1) From $\det(I + s^2A) > 0$ for all real s, condition (b) follows.

(2) From conditions (a) and (b) of this theorem, it follows that $\det(I + s^2A) > 0$ for all real s.

We prove (1) first. Since

$$\det(I + s^2A) = 1 + s^2a_1 + \dots + s^{2(n-1)}a_{n-1} + s^{2n}a_n$$

must be positive for arbitrarily large s^2, condition (b) follows.

We now prove (2). Supposing condition (b) satisfied, it is clear that for s^2 greater than or equal to some positive number s_o^2 we have

$$\det(I + s^2 A) > 0 \quad .$$

Suppose now that for some lesser number s_1^2 we have

$$\det(I + s_1^2 A) = 0 \quad .$$

Then the whole n-dimensional space is mapped by $I + s_1^2 A$ onto a linear subspace M of dimension at most $n-1$. Under the transformation $I + s_1^2 A$, the boundary $\overline{C}$ of C is mapped into a set $\overline{C}_1$. Consider the location of $\overline{C}_1$. Since A is boundary-extending for C, $\overline{C}_1$ cannot have points in common with the interior of C. On the other hand, $\overline{C}_1$ lies within the image C_o of C under $I + s_o^2 A$. For $I + s_o^2 A$ expands C (we need only take $I + s_o^2 A$ to be the transformation A of Lemma 6), and

$$(I + s_1^2 A)x = \frac{s_1^2}{s_o^2}(I + s_o^2 A)x + \left(\frac{s_o^2 - s_1^2}{s_o^2}\right)x \quad ,$$

which shows that because of the convexity of C_o, when x is in $\overline{C}$, its image under $I + s_1^2 A$ is in C_o.

Geometrical considerations, coupled with this information regarding the location of $\overline{C}_1$, show that $\overline{C}_1$ is a convex cone in the lower dimensional subspace M.

But then $\overline{C}_1$ must lie on the negative side of the supporting hyperplane to C at some non-vanishing boundary vector x. Since A is boundary-extending for C, we must have

$$(I + s_1^2 A)x = 0 \quad .$$

Substitution of this in the preceding equation shows that

$$(I + s_o^2 A)x = -\left(\frac{s_o^2 - s_1^2}{s_1^2}\right)x \quad .$$

The scalar factor multiplying x on the right-hand side of this relation is negative. However, the transformation $I + s_o^2 A$ is known to expand C and so cannot send a boundary vector of C into a vector in the opposite direction. This proves the theorem.

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